Discrete Mathematics 247 (2002) 1–12
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Resolvable perfect Mendelsohn designs with block size five

R.J.R. Abel[a], F.E. Bennett[b,*,1], G. Ge[c,2]

[a] School of Mathematics, University of New South Wales, Kensington, N.S.W. 2033, Australia
[b] Department of Mathematics, Mount Saint Vincent University, Halifax, Nova Scotia, Canada B3M 2J6
[c] Department of Mathematics, Suzhou University, Suzhou 215006, People's Republic of China

Received 6 October 1999; revised 20 October 2000; accepted 19 February 2001

Abstract

A necessary condition for the existence of a resolvable $(v,5,1)$-perfect Mendelsohn design is $v\equiv 0 \pmod 5$. This condition is shown to be sufficient for $v\geqslant 215$, with two known exceptions plus at most 17 possible exceptions below this value.

MSC: primary 05B05

Keywords: Perfect Mendelsohn design; Resolvable perfect Mendelsohn design; Resolvable holey perfect Mendelsohn design

1. Introduction

If $A=(a_1,a_2,\ldots,a_k)$ is an ordered k-tuple, then the ordered pair (a_i,a_j) is said to be *t-apart* in A if $j-i\equiv t$ modulo k.

Let v, k and λ be positive integers. A (v,k,λ)-*Mendelsohn design*, denoted briefly by (v,k,λ)-MD, is a pair $(X,\mathbf{B})$ where X is a v-set (of *points*) and $\mathbf{B}$ is a collection of ordered k-tuples (called *blocks*) such that (1) no $B\in\mathbf{B}$ contains any element of X more than once and (2) for any $x_1,x_2\in X$, $x_1\neq x_2$, the ordered pair (x_1,x_2) is 1-apart in exactly λ blocks of $\mathbf{B}$. If (2) is replaced by the stronger condition that for any t, $1\leqslant t\leqslant k-1$, every ordered pair (x_1,x_2) of distinct points appears t-apart in exactly λ blocks of $\mathbf{B}$, then the (v,k,λ)-MD is called a *perfect Mendelsohn design* and denoted briefly by (v,k,λ)-PMD.

* Corresponding author.
E-mail address: frank.bennett@msvu.ca (F.E. Bennett).
[1] Supported by NSERC Grant OGP 0005320.
[2] Supported in part by YNSFC Grant 10001026.

0012-365X/02/$ - see front matter
PII: S0012-365X(01)00157-1

In graph-theoretic terms, a (v,k,λ)-PMD is equivalent to a decomposition of the complete directed multigraph $\lambda\mathrm{DK}_v$ on v vertices into k-circuits such that for any r, $1 \leqslant r \leqslant k-1$, and for any two distinct vertices x and y there are exactly λ circuits along which the (directed) distance from x to y is r.

If we replace each k-tuple $(a_1,a_2,\ldots,a_k)$ in a (v,k,λ)-PMD by an unordered set $\{a_1,a_2,\ldots,a_k\}$, then a $(v,k,1)$-PMD becomes a *balanced incomplete block design* with parameters v, k and $\lambda = k-1$, briefly denoted by $(v,k,k-1)$-BIBD. Therefore, we can consider perfect Mendelsohn designs as a generalization of balanced incomplete block designs with $\lambda = k-1$. Mendelsohn [11] first introduced the concept of PMDs and called them perfect cyclic designs. This concept has been further studied by various authors, including Hsu and Keedwell [9] where the designs were called perfect Mendelsohn designs. We have since adopted this terminology.

If a set of blocks contains every point exactly once, then the set is called a *parallel class*. A design is called *resolvable*, if all the blocks can be partitioned into parallel classes. A resolvable (v,k,λ)-PMD is denoted by (v,k,λ)-RPMD.

It is easy to see that a necessary condition for the existence of a (v,k,λ)-RPMD is the following:

$$v \equiv 0 \pmod{k}.$$

For $k=3$, the problem of existence of $(v,3,1)$-RPMDs has been completely settled. We have the following result.

Theorem 1.1 (Bermond et al. [6]). *A necessary condition for the existence of a $(v,3,1)$-RPMD, namely, $v \equiv 0 \pmod 3$, is also sufficient, except for the non-existing design $(6,3,1)$-RPMD.*

We also have the following result for $(v,4,1)$-RPMDs.

Theorem 1.2 (Zhang [15]). *A necessary condition for the existence of a $(v,4,1)$-RPMD is $v \equiv 0 \pmod 4$. This condition is sufficient, except for $v = 4, 8$ and possibly for 49 other values, the largest of which is 336.*

In this paper, we shall focus our attention on the problem of existence of $(v,5,1)$-RPMDs. It is shown that the necessary condition for the existence of a $(v,5,1)$-RPMD, namely, $v \equiv 0 \pmod 5$, is also sufficient for $v \geqslant 215$, with 2 known exceptions and at most 17 possible exceptions below this value.

2. Auxiliary designs

In this section, we shall define some terminology and describe some of the auxiliary designs to be used in our constructions. For more detailed information on some of these related combinatorial structures, the reader is referred to [7,14].

Let $\mathrm{DK}_{n_1,n_2,\ldots,n_h}$ be the complete multipartite directed graph with vertex set $X=\bigcup_{1\leqslant i\leqslant h} X_i$, where X_i $(1\leqslant i\leqslant h)$ are disjoint sets with $|X_i|=n_i$, $v=\sum_{1\leqslant i\leqslant h} n_i$, and where two vertices x and y from different sets X_i and X_j are joined by exactly one arc from x to y and one arc from y to x. A *holey perfect Mendelsohn design* (HPMD) with block size k is an order pair $(X,\mathbf{A})$ where $\mathbf{A}$ is a set of k-circuits (directed cycles of length k), called *blocks*, which form an arc-disjoint decomposition of $\mathrm{DK}_{n_1,n_2,\ldots,n_h}$ with the property that for any integer r $(1\leqslant r\leqslant k-1)$ and any two vertices x and y from different sets X_i and X_j, there is exactly one circuit $c\in\mathbf{A}$ such that the directed distance along c from x to y is r. Each X_i $(1\leqslant i\leqslant h)$ is called a *hole* (or *group*) of the design and the multiset $\{n_1,n_2,\ldots,n_h\}$ is called the *type* of the design. We denote the design by $(v,k,1)$-HPMD (or briefly k-HPMD) and use an "exponential" notation to describe its type in general: a type $1^i2^r3^k\ldots$ denotes i occurrences of $1, r$ occurrences of 2, etc.

A k-HPMD is said to be *resolvable* if its blocks can be divided into parallel classes (a parallel class is collection of blocks which form a partition of the point set, X). We denote it by k-RHPMD.

A k-HPMD is called a *frame* k-PMD if its blocks can be divided into partial parallel classes, each of which partitions $X\backslash X_i$ for some group X_i. We use k-FPMD to denote it.

We also make use of certain designs used in standard design theory, for instance group divisible design, transversal design with block size k and group size n, denoted by $\mathrm{TD}(k,n)$ and pairwise balanced design on v points and with block sizes from the set K, denoted by (v,K)-PBD. For the definition of these standard designs, the reader is referred to [8]. When resolvable, these designs are given by the prefix R.

3. Direct constructions

The constructions used in this paper will combine both direct and recursive methods. Finite fields and abelian groups play an important role in our direct constructions. For most of our direct constructions, we adopt the familiar method using difference sets as in the construction of BIBDs, where we use finite abelian groups to generate the set of blocks for a given design. That is, instead of listing all the blocks of the design, we shall list a set of base blocks and generate the others by an additive group and perhaps some further automorphisms. If G is the additive group under consideration, then we shall adopt the following convention:

$$\operatorname{dev}\mathbf{B}=\{B+g\colon B\in\mathbf{B} \text{ and } g\in G\},$$

where $\mathbf{B}$ is the collection of base blocks of the design.

Lemma 3.1. *For* $p=7,11,13,17,19,23,25,27,29,37,41,47$, *there exists a* 5-*FPMD of type* 5^p.

Proof. Let the point set be $Z_5 \times \mathrm{GF}(p)$. We need two initial base blocks of the form

$$B_1 = ((0,a),\ (1,b),\ (4,c),\ (4,d),\ (1,e)),$$
$$B_2 = ((0,f),\ (2,g),\ (3,h),\ (3,i),\ (2,j)).$$

We then multiply these by $(1, y)$ for y a quadratic residue in $\mathrm{GF}(p)$ to obtain $p-1$ blocks. These $p-1$ blocks will form a partial parallel class missing the group $Z_5 \times \{0\}$. For other blocks, we cycle these mod$(5, p)$. We list the corresponding two initial base blocks for each p below. For $p = 25$ and 27, x is a primitive element of $\mathrm{GF}(p)$ satisfying $x^2 = x + 3$ and $x^3 = x + 2$ respectively.

$p = 7$: $((0,4),\ (1,5),\ (4,6),\ (4,1),\ (1,2))$,
$((0,3),\ (2,6),\ (3,5),\ (3,1),\ (2,4))$;

$p = 11$: $((0,7),\ (1,8),\ (4,2),\ (4,5),\ (1,3))$,
$((0,4),\ (2,3),\ (3,9),\ (3,8),\ (2,7))$;

$p = 13$: $((0,2),\ (1,3),\ (4,7),\ (4,9),\ (1,8))$,
$((0,3),\ (2,1),\ (3,7),\ (3,10),\ (2,11))$;

$p = 17$: $((0,1),\ (1,4),\ (4,6),\ (4,16),\ (1,12))$,
$((0,10),\ (2,5),\ (3,6),\ (3,2),\ (2,15))$;

$p = 19$: $((0,1),\ (1,8),\ (4,6),\ (4,15),\ (1,17))$,
$((0,8),\ (2,14),\ (3,10),\ (3,5),\ (2,9))$;

$p = 23$: $((0,9),\ (1,5),\ (4,13),\ (4,22),\ (1,3))$,
$((0,7),\ (2,4),\ (3,13),\ (3,10),\ (2,20))$;

$p = 25$: $((0,x^8),\ (1,x^1),\ (4,x^{12}),\ (4,x^{11}),\ (1,x^{10}))$,
$((0,x^{19}),\ (2,x^5),\ (3,x^8),\ (3,x^{17}),\ (2,x^{18}))$;

$p = 27$: $((0,x^{12}),\ (1,x^6),\ (4,1),\ (4,x^{13}),\ (1,x^3))$,
$((0,x^{25}),\ (2,x^{22}),\ (3,x^{18}),\ (3,x^{21}),\ (2,x^{17}))$;

$p = 29$: $((0,16),\ (1,11),\ (4,26),\ (4,7),\ (1,6))$,
$((0,18),\ (2,4),\ (3,15),\ (3,6),\ (2,2))$;

$p = 37$: $((0,1),\ (1,5),\ (4,17),\ (4,12),\ (1,16))$,
$((0,5),\ (2,19),\ (3,25),\ (3,24),\ (2,27))$;

$p = 41$: $((0,6),\ (1,7),\ (4,11),\ (4,32),\ (1,40))$,
$((0,25),\ (2,3),\ (3,6),\ (3,23),\ (2,32))$;

$p = 47$: $((0,2),\ (1,3),\ (4,12),\ (4,22),\ (1,29))$,
$((0,40),\ (2,4),\ (3,9),\ (3,30),\ (2,10))$. □

Lemma 3.2. *There exists a* 5*-FPMD of type* 5^6.

Proof. Let the point set be Z_{30}. We list the base blocks below; Other blocks in the FPMD are obtained from them using the '+2 method' which means that only multiples of 2 should be added to each base block;

$$(1,3,2,10,17),\quad (1,14,23,3,4),\quad (1,8,22,5,9),\quad (1,17,26,22,3),$$
$$(1,9,4,2,5),\quad (0,21,8,10,11),\quad (0,17,22,26,15),\quad (0,5,16,2,27),$$
$$(0,15,5,28,20),\quad (0,29,26,16,9).$$

For $t\in\{0,2,4\}$, adding $t,t+6,t+12,t+18,t+24$ to any of the first 5 base blocks produces a partial parallel class with hole $\{t,t+6,t+12,t+18,t+24\}$. Similarly, adding $t,t+6,t+12,t+18,t+24$ to any of the last five blocks gives a partial parallel class with hole $\{t+1,t+7,t+13,t+19,t+25\}$. □

Lemma 3.3. *For* $n=8,9,10,12,14,15$, *there exists a* 5*-FPMD of type* 5^n.

Proof. Let the point set be $Z_{5(n-1)}\cup X$, where $X=\{x_1,x_2,x_3,x_4,x_5\}$. Also, let X and $\{j,(n-1)+j,2(n-1)+j,3(n-1)+j,4(n-1)+j\}$ for $j=0,1,\dots,n-2$ be the holes of the 5-FPMD. Below are the required base blocks to be developed mod $5(n-1)$:

$n=8$: $(0,19,11,17,13)$, $(33,16,24,29,32)$, $(17,2,27,5,15)$,
$(x_1,6,10,22,11)$, $(x_2,30,4,20,31)$, $(x_3,3,26,8,23)$,
$(x_4,1,34,25,19)$, $(x_5,18,13,12,9)$;

$n=9$: $(0,9,3,21,2)$, $(34,5,36,39,14)$, $(21,31,38,35,25)$,
$(11,23,22,28,10)$, $(x_1,33,12,26,13)$, $(x_2,4,9,2,19)$,
$(x_3,37,20,6,1)$, $(x_4,29,18,3,7)$, $(x_5,27,15,17,30)$;

$n=10$: $(0,44,41,33,22)$, $(40,26,11,12,5)$, $(41,43,1,8,30)$,
$(6,10,23,21,31)$, $(39,35,19,38,33)$, $(x_1,22,2,17,25)$,
$(x_2,4,16,42,14)$, $(x_3,15,29,34,28)$, $(x_4,44,32,3,24)$,
$(x_5,13,37,20,7)$;

$n=12$: $(0,29,42,28,16)$, $(6,31,7,27,26)$, $(29,21,14,52,50)$,
$(23,25,39,48,5)$, $(54,16,46,40,9)$, $(35,30,17,38,20)$,
$(45,51,15,41,13)$, $(x_1,34,37,28,32)$, $(x_2,18,3,4,12)$,
$(x_3,8,36,43,24)$, $(x_4,19,42,47,2)$, $(x_5,1,53,49,10)$;

$n=14$: $(0,4,27,48,51)$, $(5,64,4,50,7)$, $(33,2,62,48,58)$,
$(43,19,12,42,54)$, $(24,1,34,17,45)$, $(44,61,20,29,56)$,
$(53,37,36,51,18)$, $(31,21,3,9,46)$, $(22,38,57,28,30)$,
$(x_1,59,32,63,27)$, $(x_2,6,49,60,40)$, $(x_3,25,16,23,41)$,
$(x_4,47,55,10,35)$, $(x_5,14,15,11,8)$;

$n=15$: $(0,29,7,68,6)$, $(48,11,61,38,2)$, $(64,1,16,69,20)$,
$(30,19,4,15,7)$, $(25,44,9,60,55)$, $(47,45,13,6,63)$,
$(22,24,23,32,49)$, $(58,62,31,41,68)$, $(52,57,39,5,27)$,
$(43,10,26,50,46)$, $(x_1,33,12,18,36)$, $(x_2,17,37,8,66)$,
$(x_3,40,53,65,21)$, $(x_4,51,54,29,59)$, $(x_5,3,34,35,67)$.

For each of these designs, adding $5,10,15,\ldots,5(n-2) \bmod (5(n-1))$ to the first base block produces a partial parallel class missing $\{x_1,x_2,\ldots,x_5\}$. The other base blocks form a partial parallel class which misses the hole $\{0,(n-1),2(n-1),3(n-1),4(n-1)\}$. □

The following lemma is given in [4].

Lemma 3.4. *If $p \equiv 1 \pmod 5$ is a prime power, then there exists a 5-FPMD of type 1^p.*

Proof. Take the point set as GF(p); the base blocks to be developed over GF(p) are $(b,\ w\cdot b, w^2\cdot b, w^3\cdot b, w^4\cdot b)$ for w a given 5th root of unity and $b\in\{1,x,x^2,\ldots,x^{(p-6)/5}\}$ where x is a primitive element in GF(p). These base blocks form the holey parallel class missing the point 0. □

Lemma 3.5. *There exists a 5-FPMD of type 1^{21}.*

Proof. This construction comes from [8, p. 117]. Let the point set be Z_{21} and cycle the following 4 blocks mod 21. These blocks form the holey parallel class missing the point 0:

$$(8,9,11,14,18),\quad (16,5,1,19,17),\quad (20,4,10,3,12),\quad (2,15,6,13,7). \qquad \square$$

Lemma 3.6. *For $v=36,46$, there exists a 5-FPMD of type 1^v.*

Proof. Let the point set be $Z_{v-1}\cup\{x\}$, and let $\{\{j\}: j=0,1,\ldots,v-2\}\cup\{x\}$ be the hole set of the 5-FPMD. Below are the required base blocks to be cycled mod $v-1$. In both cases, the last block generates just $(v-1)/5$ blocks which form the partial parallel class missing the point x. The other base blocks form the partial parallel class missing the point 0:

$v=36$: $(29,6,10,24,19)$, $(22,33,14,11,31)$, $(30,26,18,1,2)$,
$(34,8,27,9,12)$, $(28,16,5,7,13)$, $(21,20,25,15,23)$,
$(x,4,17,3,32)$, $(0,7,14,21,28)$;

$v=46$: $(4,20,23,8,15)$, $(40,19,29,5,32)$, $(27,28,26,43,21)$,
$(24,16,30,14,2)$, $(38,18,13,12,6)$, $(33,7,35,25,36)$,
$(31,1,42,10,22)$, $(11,37,41,34,9)$, $(x,39,44,17,3)$,
$(0,36,27,18,9)$. □

Lemma 3.7. *For* $n = 5, 7, 8, 9, 10, 12, 14, 15$, *there exists a* 5-*RHPMD of type* 5^n.

Proof. For $n = 5$, this design is obtained by constructing a $(5,1)$-RGDD of type 5^5 and forming a $(5,5,1)$-RPMD on each of its blocks. For the others, let the point set be $Z_{5(n-1)} \cup X$ where $X = \{x_1, x_2, x_3, x_4, x_5\}$; also, let X and $\{j, (n-1)+j, 2(n-1)+j, 3(n-1)+j, 4(n-1)+j\}$ for $j = 0, 1, \ldots, n-2$ be the holes of the 5-RHPMD. The required base blocks, which form a parallel class are given below; cycle each of them mod $5(n-1)$:

$n = 7$: $(0,4,26,27,13)$, $(3,23,16,24,1)$, $(x_1,11,14,9,22)$,
$(x_2,21,5,2,7)$, $(x_3,6,25,10,20)$, $(x_4,28,19,18,29)$,
$(x_5,12,8,17,15)$;

$n = 8$: $(0,19,1,2,13)$, $(21,23,29,4,24)$, $(30,11,6,33,31)$,
$(x_1,14,17,5,9)$, $(x_2,3,8,32,28)$, $(x_3,18,27,7,15)$,
$(x_4,26,20,10,22)$, $(x_5,25,16,34,12)$;

$n = 9$: $(0,29,3,1,12)$, $(13,26,15,2,33)$, $(19,37,39,34,9)$,
$(31,17,36,30,26)$, $(x_1,23,6,18,25)$, $(x_2,21,22,35,28)$,
$(x_3,24,5,11,20)$, $(x_4,10,7,32,14)$, $(x_5,27,4,8,38)$;

$n = 10$: $(0,34,11,28,12)$, $(15,30,29,32,25)$, $(39,40,42,1,9)$,
$(13,38,21,35,6)$, $(31,43,19,5,44)$, $(x_1,8,4,27,33)$,
$(x_2,14,24,3,16)$, $(x_3,36,2,7,26)$, $(x_4,41,22,17,37)$,
$(x_5,23,20,18,10)$;

$n = 12$: $(0,32,1,31,37)$, $(5,36,39,34,26)$, $(19,21,9,17,14)$,
$(12,16,52,46,28)$, $(30,44,24,53,40)$, $(45,43,42,8,25)$,
$(23,51,15,11,38)$, $(x_1,49,3,29,22)$, $(x_2,18,41,2,27)$,
$(x_3,47,4,50,33)$, $(x_4,10,20,35,48)$, $(x_5,13,54,6,7)$;

$n = 14$: $(0,41,34,64,11)$, $(18,45,26,21,20)$, $(47,2,53,33,62)$,
$(37,61,19,25,3)$, $(63,40,48,31,4)$, $(42,9,58,7,54)$,
$(15,24,43,13,5)$, $(6,23,59,56,16)$, $(22,29,60,39,50)$,
$(x_1,49,17,8,12)$, $(x_2,52,55,51,1)$, $(x_3,36,57,14,32)$,
$(x_4,30,35,10,38)$, $(x_5,27,28,44,46)$;

$n = 15$: $(0,19,22,18,11)$, $(1,35,12,62,25)$, $(47,49,64,32,68)$,
$(41,33,44,4,28)$, $(39,29,55,5,34)$, $(53,50,23,6,14)$,
$(42,9,63,67,24)$, $(54,61,16,15,59)$, $(58,43,65,66,13)$,
$(20,36,45,26,38)$, $(x_1,8,56,27,21)$, $(x_2,52,17,48,46)$,
$(x_3,31,37,69,57)$, $(x_4,7,30,40,10)$, $(x_5,3,60,51,2)$. □

4. Recursive constructions

To obtain our main results, we shall use the following basic constructions. These constructions are similar to those used in [12,13,5].

Construction 4.1 (Weighting). *Let* $(\mathscr{X},\mathscr{G},\mathscr{A})$ *be a GDD, and let* $w:\mathscr{X}\to Z^{+}\cup\{0\}$ *be a weight function on* $\mathscr{X}$. *Suppose that for each block* $A\in\mathscr{A}$, *there exists a k-FPMD of type* $\{w(x)\colon x\in A\}$. *Then there is a k-FPMD of type* $\{\sum_{x\in G_i} w(x)\colon G_i\in\mathscr{G}\}$.

Construction 4.2 (Inflating RHPMDs by RTDs). *If there exists a k-RHPMD of type* h^u *and an* RTD(k,m), *then there exists a k-RHPMD of type* $(mh)^u$.

Construction 4.3 (Inflating FPMDs by RTDs). *If there exists a k-FPMD of type* h^u *and an* RTD(k,m), *then there exists a k-FPMD of type* $(mh)^u$.

Construction 4.4 (Filling in holes). *Suppose there is a k-FPMD with type* $T=\{t_i\colon i=1,2,\ldots,n\}$ *and let* $a\geqslant 0$. For $i=1,2,\ldots,n$, *suppose there is a k-FPMD with type* $T_i\cup\{a\}$, *where* $\sum_{t\in T_i} t=t_i$. *Then there is a k-FPMD with type* $(\bigcup_{i=1}^{n} T_i)\cup\{a\}$.

Construction 4.5 (Constructions using frames). *Suppose there is a k-FPMD with type* $T=\{t_i\colon i=1,2,\ldots,n\}$ *and let* $a>0$. *For* $i=1,2,\ldots,n$, *suppose there is a k-RHPMD with type* $T_i\cup\{a\}$, *where* $\sum_{t\in T_i} t=t_i$. *Then there is a k-RHPMD with type* $(\bigcup_{i=1}^{n} T_i)\cup\{a\}$.

Construction 4.6 (Breaking up groups). *If there is a k-RHPMD of type* t^u, $t=t_1h$ *and a k-RHPMD of type* t_1^h, *then there exists a k-RHPMD of type* t_1^{uh}.

Construction 4.7. *If there exists an* RTD$(5,g)$ *and a* 5*-FPMD of type* 1^g, *then there exists a* 5*-RHPMD of type* 5^g.

Proof. Start with an RTD$(5,g)$, delete one of its parallel classes, form a $(5,5,1)$-RPMD on each block in the other parallel classes. This produces a design with $4(g-1)$ parallel classes; all these parallel classes except the last one will remain unaltered in the final design. Finally, form a 5-FPMD of type 1^g on each group of the TD. Each block (a_1,a_2,a_3,a_4,a_5) in the last of the $4(g-1)$ parallel classes above is combined with the holey parallel classes from the FPMDs missing the points $a_1,a_2,\ldots,a_5$. This produces the required 5-RHPMD(5^g) with $5(g-1)$ parallel classes; the holes in this RHPMD are the blocks initially deleted from one of the parallel classes in RTD$(5,g)$. □

In order to use the above constructions, we need the following known results on TDs.

Lemma 4.8 (Colbourn and Dinitz [8]). (1) *An* RTD$(5,m)$ *exists for all* $m>4$ *except for* $m=6$ *and possibly for* $m\in\{10,14,18,22\}$.

(2) *A* TD$(8,m)$ *exists for all* $m\geqslant 67$ *except possibly for* $m\in\{68,74,75\}$.

(3) *A* TD$(q+1,q)$ *exists when* q *is a prime power.*

5. Existence of 5-FPMD(5^n)

As shown in [12], frames are very useful for constructing resolvable designs. 5-FPMDs will play an important role in our paper to solve the $(v,5,1)$-RPMD problem. In this section, we shall prove the existence of 5-FPMDs of type 5^n.

Lemma 5.1. *There exists a* 5-*FPMD of type* 5^n *for each* $n\in S_1=\{16,21,31,36,42,43,44,46\}\cup[48,92]\cup[186,\infty)$.

Proof. For each $n\in\{16,21,36,46\}$, apply Construction 4.3 with $m=5$ to the 5-FPMD of type 1^n obtained by Lemmas 3.4–3.6. For each other given $n\in S_1$, we have (from [8]) a $(n,\{6,7,8,9\})$-PBD. This gives a $\{6,7,8,9\}$-GDD of type 1^n. Applying Construction 4.1 with weight 5, we obtain a 5-FPMD of type 5^n. The input designs come from Lemmas 3.1–3.3. The proof is complete. □

Lemma 5.2. *There exists a* 5-*FPMD of type* 5^n *for each* $n\in S_2=[72,207]$.

Proof. Start with a TD$(11,t)$ for $t\in\{11,19\}$. Truncate 5 groups of this TD$(11,t)$ to sizes a_i $(1\leqslant i\leqslant 5)$, where $a_i=0$ or $6\leqslant a_i\leqslant t$, $a_i\neq 18$. For all n in the given range we can obtain a $\{6,7,\dots,17,19\}$-GDD of type $t^6a_1^1a_2^1\cdots a_5^1$ on n points; taking $t=11,19$, respectively, handles all n in the ranges $[72,121]$ and $[120,207]$. Applying Construction 4.1 with weight 5 and filling in the holes, we then obtain the 5-FPMDs of type 5^n as desired. The input designs come from Lemmas 3.1–3.3. The proof is complete. □

Theorem 5.3. *There exists a* 5-*FPMD of type* 5^n *for each* $n\geqslant 6$ *except possibly for* $n\in F=\{18,20,22,24,26,28,30,32,33,34,35,38,39,40,45\}$.

Proof. Combine Lemmas 3.1–3.3 and 5.1–5.2, which complete the proof. □

6. Main result

In this section, we shall present the proof of our main result as mentioned before. In fact, we shall prove a slightly stronger result on RHPMDs. First, we establish a preliminary bound and then treat the small orders. Denote $U=\{n$: a 5-RHPMD of type 5^n exists$\}$. Also let F be defined as in Theorem 5.3, that is, $F=\{18,20,22,24,26,28,30,32,33,34,35,38,39,40,45\}$. We need the following working lemma.

Lemma 6.1. *Suppose* $1 \leqslant x \leqslant 4$ *and a* $\mathrm{TD}(6+x, m)$ *exists. Suppose also there exist* 5-*FPMDs of type* 5^t *for* $t = m, a_2, \ldots, a_x$ *and a* 5-RHPMD(5^{a_1+1}), *where* $0 \leqslant a_1 < m$ *and* $0 \leqslant a_i \leqslant m$ *for* $i > 1$. *If* $u = 6m + a_1 + a_2 + \cdots + a_x + 1$, *then there exists a* 5-*RHPMD of type* 5^u.

Proof. Truncate x groups in the $\mathrm{TD}(6+x, m)$ to sizes a_i, $1 \leqslant i \leqslant x$ and use a deleted point from the group of size a_1 to redefine groups. This gives a $\{6, 7, \ldots, 6+x, m, a_2, a_3, \ldots, a_x\}$-GDD with groups of sizes $6, 7, \ldots, 6+x-1$ and a_1. Apply Construction 4.1 with weight 5, add 5 infinite points and use 5-RHPMDs of types 5^7, 5^8, 5^9, 5^{10}, coming from Lemma 3.7, and type 5^{a_1+1} to fill in holes, we then obtain the design as desired. Here, we also need 5-FPMDs of types 5^n for $n \in \{6, 7, 8, 9, 10\}$ as input designs; these all come from Theorem 5.3. This completes the proof. □

Lemma 6.2. *If* $u \geqslant 457$, *then* $u \in U$.

Proof. From Lemma 4.8, we have a $\mathrm{TD}(8, m)$, a $\mathrm{TD}(8, m+1)$ or a $\mathrm{TD}(8, m+2)$ for any $m \geqslant 67$. Apply Lemma 6.1 with $m \geqslant 67$, $x = 2$, $a_1 = 6$ and $48 \leqslant a_2 \leqslant m$, we have $[6m+55, 7m+7] \subset U$. It is not difficult to check that these intervals overlap when m runs over $[67, \infty)$. This completes the proof. □

Lemma 6.3. *If* $73 \leqslant u \leqslant 456$, *then* $u \in U$.

Proof. The proof is similar to that of Lemma 6.2. Here we take $x = 4$, $m \in \{53, 49, 43, 37, 31, 29, 23, 19, 17, 13, 11, 9\}$, $a_1 = 6$, $0 \leqslant a_i \leqslant m$ and $a_i \notin F \cup \{2, 3, 4, 5\}$. We can obtain some intervals for u and by running a simple computer program, it is readily checked that these intervals overlap. The proof is complete. □

Lemma 6.4. *If* $58 \leqslant u \leqslant 72$, *then* $u \in U$.

Proof. For $65 \leqslant u \leqslant 72$, in $\mathrm{PG}(2, 8)$ there exists an oval consisting of 10 points no three of which are collinear. If we delete $t+1$ oval points ($1 \leqslant t \leqslant 8$) and take the blocks containing one of the deleted points as groups we obtain a $\{7, 8, 9\}$-GDD of type $7^t 8^{9-t}$. Giving weight 5 to the points of this GDD, we obtain a 5-FPMD of type $35^t 40^{9-t}$. Finally, add 5 infinite points and use 5-RHPMDs of types 5^n for $n = 8$ and 9. This gives the desired 5-RHPMDs of types 5^{73-t} for $1 \leqslant t \leqslant 8$. For $58 \leqslant u \leqslant 64$, delete $t+1$ oval points ($0 \leqslant t \leqslant 6$) from $\mathrm{AG}(2, 8)$ to obtain a $\{6, 7, 8\}$-GDD of type $6^t 7^{9-t}$, a 5-FPMD of type $30^t 35^{9-t}$, and finally, by adding 5 infinite points, 5-RHPMDs of types 5^{64-t} for $0 \leqslant t \leqslant 6$. □

Lemma 6.5. *If* $50 \leqslant u \leqslant 57$, *then* $u \in U$.

Proof. The proof is similar to the one for the previous lemma. Here, we delete $t+1$ oval points ($0 \leqslant t \leqslant 7$) from $\mathrm{PG}(2, 7)$ to obtain a $\{6, 7, 8\}$-GDD of type $6^t 7^{8-t}$, a 5-FPMD of type $30^t 35^{8-t}$ and finally (by adding 5 infinite points and using 5-RHPMDs of types 5^7 and 5^8) 5-RHPMDs of types 5^{57-t} for $0 \leqslant t \leqslant 7$. □

Lemma 6.6. *If* $u \in \{43, 44, 47, 48, 49\}$, *then* $u \in U$.

Proof. Start with a TD(8, 7) and delete 7 points from a block so as to form a $\{7, 8\}$-GDD of type $6^7 7^1$. For $u = 44, 48$, delete six or two points from one of the groups of size 6 to give $\{6, 7, 8\}$-GDDs of types $6^6 7^1$ and $6^6 7^1 4^1$. For $u = 49, 47, 43$, delete 1, 3 or 7 points from the group of size 7 to give $\{6, 7, 8\}$-GDDs of types 6^8, $6^7 4^1$ and 6^7. Give weight 5 to the points of these GDDs to obtain 5-FPMDs of types $30^6 35^1$, $30^6 35^1 20^1$, 30^8, $30^7 20^1$ and 30^7. Adding 5 infinite points and using 5-RHPMDs of types 5^n for $n = 5, 7, 8$ gives the desired 5-RHPMDs of types 5^u for $u \in \{43, 44, 47, 48, 49\}$. □

Lemma 6.7. *If* $u \in \{13, 17, 21, 25, 29, 33, 37, 41\}$, *then* $u \in U$.

Proof. For each given u, a 5-RGDD of type 5^u exists from [1] (for $u \neq 37$) and [2] (for $u = 37$); hence there exists a 5-RHPMD of the same type. □

Lemma 6.8. *If* $u \in \{11, 16, 31, 36, 46\}$, *then* $u \in U$.

Proof. For $u = 11, 16, 31, 36, 46$, a 5-FPMD of type 1^u exists by either Lemma 3.4 or Lemma 3.6. Applying Construction 4.7 now gives the results as desired. □

Lemma 6.9. *If* $u \in \{35, 40, 45\}$, *then* $u \in U$.

Proof. From Lemma 3.7, we have 5-RHPMDs of type 5^n for $n = 7, 8, 9$. Inflating by 5 and filling in holes with a 5-RHPMD of type 5^5, (which exists by Lemma 3.7) we obtain the 5-RHPMDs as desired. □

Now, we are in a position to state our main result.

Theorem 6.10. *A necessary condition for existence of a* $(v, 5, 1)$-*RPMD is* $v \equiv 0 \pmod 5$. *This condition is also sufficient except for* $v \in \{10, 15\}$ *and possibly for* $v \in \{20, 30, 90, 95, 100, 110, 115, 120, 130, 135, 140, 150, 160, 170, 190, 195, 210\}$.

Proof. The nonexistence of a $(10, 5, 1)$-PMD has been shown in [3]. Hence, there is no $(10, 5, 1)$-RPMD. Also, in [10], it is shown that there is no $(15, 5, 4)$ RBIBD; hence a $(15, 5, 1)$-RPMD cannot exist. The conclusion then follows by the combination of Lemmas 6.2–6.9 and 3.7. □

Acknowledgements

The first author would like to thank the Department of Mathematics at Mount Saint Vincent University for providing computer services which were used to obtain some of the direct constructions in this paper. The second author would like to acknowledge the support of the Natural Sciences and Engineering Research Council of Canada under Grant OGP 0005320 and a grant from Mount Saint Vincent University Committee on Research and Publications. A portion of this research was carried out while the third author was visiting Mount Saint Vincent University in 1999, and he wishes to express many thanks to the Department of Mathematics for their services.

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